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FORMULATION AND SOLUTION OF THE PROBLEM OF THE STABILITY OF A COHESIONLESS CHANNEL BOTTOM

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The problem of the stability of the sandy bottom of a rectangular channel with respect to spatially one-dimensional perturbations is formulated. The channel stability problem is solved using the sediment discharge formula modified by taking into account the effect of free-surface perturbations on sediment transport. Analytic dependences of the propagation velocity of bottom perturbations on time are obtained for small Froude numbers. Wavelengths for the most rapidly growing bottom perturbations are obtained analytically, and the problem of the evolution of a single bottom perturbation is solved.

Key words: *sediment discharge, sandy bottom of channel, channel stability.*

Introduction. In the present paper, the sediment movement theory proposed in [1, 2] is elaborated, the sediment discharge formula is refined, and one of the fundamental problems of channel dynamics — the channel stability problem — is solved in linearized form. We note that attempts to solve the specified problem using phenomenological models of sediment movement have been undertaken in many studies dealing with various mechanisms of sediment movement in channels (see, for example, [3–12]). In the present work, the previously obtained equations of channel deformations were used and generalized. This made it possible to obtain analytical dependences of the propagation velocity of bottom perturbations on time at small Froude numbers, to determine the wavelength for the most rapidly growing bottom perturbations, and to solve the problem of motion and evolution of single bottom perturbation in analytic form.

1. Equations of Hydrodynamics. The problem of the erosion of a sandy river bed or channel by turbulent fluid flow is considered. To determine the depth-averaged fluid velocity field, the standard hydraulic equations [12, 13], called the equations of the plan problem. These hydrodynamic equations in the shallow water approximation take into account the dependence of mass forces on the average channel slope and the quadratic resistance law. In addition, the equations include viscous stress with the turbulent exchange coefficient. Since the bottom profile changes slowly, stationary equations of motion are used. Equations which are one-dimensional on the coordinate x are considered. The x direction coincides with the depth-averaged flow velocity direction z, and the Cartesian coordinate $V(x)$ is directed vertically upward (Fig. 1). Let $\zeta(t,x)$ be the bed level, $\eta(t,x) = h + \zeta(t,x)$ the free-surface level, and h the average channel depth. The coordinate origin of the channel is chosen so that the functions $\zeta(x)$ and $\eta(x)$ averaged over the x coordinate are equal to zero: $\langle\zeta(x)\rangle = 0$ and $\langle\eta(x)\rangle = 0$. The equation of motion is written as

$$V\,\frac{\partial V}{\partial x} = gJ - g\,\frac{\partial \eta}{\partial x} - \frac{\lambda V^2}{h+\eta-\zeta} + \nu_t\,\frac{\partial^2 V}{\partial x^2}.$$

This equation differs from the one adopted in hydraulics [12] by the presence of the term $\nu_t\,\partial^2 V/\partial x^2$. A similar term is included in the Navier–Stokes equation, which, instead of the turbulent exchange coefficient ν_t, contains the significantly smaller molecular viscosity coefficient ν. For the wavelengths exceeding or comparable to the depth h,

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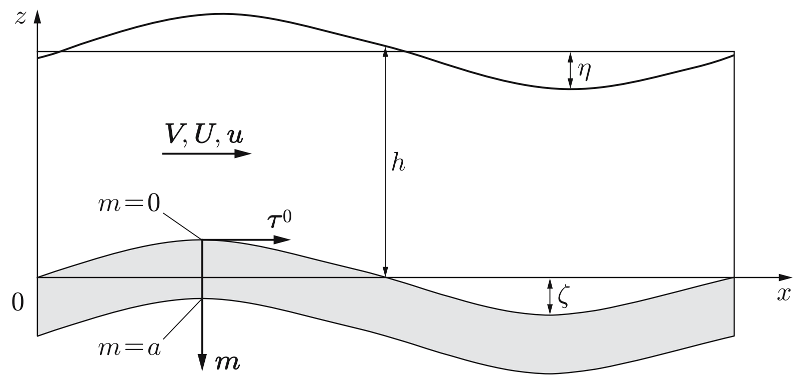

Fig. 1. Diagram of the problem.

the contribution of this term is negligibly small; therefore, it can be ignored. In stability theory, this term plays an important role because it limits the growth of the amplitude of very short waves.

We supplement the equation of motion with the equation of conservation of the discharge Q:

$$\frac{1}{2}\frac{\partial V^2}{\partial x} - gJ + g\frac{\partial \eta}{\partial x} + \frac{\lambda V^3}{Q} - \nu_t \frac{\partial^2 V}{\partial x^2} = 0; \tag{1.1}$$

$$Q = (h + \eta - \zeta)V. \tag{1.2}$$

Here J is the average channel slope, λ is the hydraulic resistance coefficient, and g is the acceleration due to gravity. To solve the stability problem, it is necessary to linearize these equations. In the case of an even bottom ($\zeta = 0$), the solution of Eqs. (1.1) and (1.2) is flow at velocity U which has an even free surface $\eta = 0$. The velocity U is determined from the balance between the resistance forces and the product of the bottom slope into the gravity:

$$U = Q/h = \sqrt{ghJ/\lambda}. \tag{1.3}$$

We linearize Eqs. (1.1) and (1.2) over small perturbations $u = V - U$, η, and ζ. Then, the expressions for different powers of V are written as

$$V = \frac{Q}{h + \eta - \zeta} = \frac{Q}{h} + \frac{Q}{h^2}(\zeta - \eta),$$

$$V^2 = \frac{Q^2}{h^2} + 2\frac{Q^2}{h^3}(\zeta - \eta), \qquad V^3 = \frac{Q^3}{h^3} + 3\frac{Q^3}{h^4}(\zeta - \eta).$$

Substitution of these expressions into Eqs. (1.1) with the use of (1.3) yields

$$\frac{\mathrm{Fr} - 1}{\mathrm{Fr}}\frac{\partial \eta}{\partial x} - \frac{\partial \zeta}{\partial x} + \frac{3\lambda}{h}(\eta - \zeta) - h\nu_q \frac{\partial^2(\eta - \zeta)}{\partial x^2} = 0,$$
$$\mathrm{Fr} = \frac{U^2}{gh}, \qquad \nu_q = \frac{\nu_t}{Q}. \tag{1.4}$$

Here Fr and ν_q are the dimensionless Froude number and the turbulent exchange coefficient.

2. Equation of Erosion of the Bottom Surface and the Sediment Discharge Formula. The variation in the bed level can be found from the mass conservation equation for sand [12]:

$$\frac{\partial \zeta}{\partial t} + \frac{1}{\rho_s(1 - \varepsilon)}\frac{\partial G}{\partial x} = 0, \tag{2.1}$$

where ρ_s is the density of sand particles and ε is the porosity coefficient of the sandy bottom.

To close the system of equations, it is necessary to have the dependence of the sediment discharge vector $\boldsymbol{G}$ on the fluid stress vector on the bottom and the local slopes of the bottom and free surfaces.

Previous calculations of the sediment discharge vector ignored the variation in the free surface [1, 2]. The model of a water–soil mixture moving in a thin layer near the bottom is formulated. The rheological law of the mixture takes into account the Coulomb law for particle friction and the Prandtl turbulent friction for the fluid. In [1, 2], solution of the boundary-value problem for a moving layer of particles yielded the sediment discharge formula

$$G = G_0 V^3 \Big(1 + \frac{1}{\tan\varphi}\frac{\partial\zeta}{\partial x}\Big)^{-1}; \tag{2.2}$$

$$G_0 = \frac{4}{3}\,\frac{\rho_s\rho_w\lambda^{3/2}}{\varkappa(\rho_s-\rho_w)g\tan\varphi} \tag{2.3}$$

(ρ_w is the density of water, $\varkappa$ is the Kármań constant, and φ is the internal friction angle of sand). In the case of an even bottom, formula (2.2) is similar to the most widely used empirical Meyer-Peter and Müller formula (see [14]), which can be obtained by integrating the depth distribution of particles given in [15]. However, these formulas are insufficient for solving the channel stability problem. One needs the dependence of (2.2) on the function $\zeta(t,x)$ which defines the bottom surface relief, which can be obtained by methods of continuum mechanics.

Thus, Eqs. (1.1), (1.2), and (2.1)–(2.3) form a closed system of equations. The obtained model is consistent with the standard empirical theories extended to the case of a two-dimensional bottom surface and adequately describes the erosion of the banks of an inclined channel [1, 2]. However, in the problem of the stability of a flat bottom surface, the perturbations of the bottom and free surfaces are comparable in value. If the additional term in the equation of motion is ignored, small perturbations of the horizontal bottom surface decrease with time and dunes do not form, whereas accounting for this term allows one to describe regimes in which bottom perturbations with certain wavelengths increase with time.

3. Refining the Sediment Discharge Formula. A model for a water-soil mixture in a thin bottom layer was proposed in [1]. The mathematical formulation of the problem for the specific mass discharge of solid particles moving in the active bottom layer includes the following equations:

$$\frac{\partial p}{\partial m} = \rho g, \qquad \rho = f\rho_s - (1-f)\rho_w; \tag{3.1}$$

$$\frac{\partial p}{\partial x} + \rho g\,\frac{\partial\zeta}{\partial x} + \frac{\partial\tau}{\partial m} = 0; \tag{3.2}$$

$$\tau = -(\tau_s+\tau_w)\,\frac{\partial w}{\partial m}\Big/\Big|\frac{\partial w}{\partial m}\Big| = \tau_s + \tau_w; \tag{3.3}$$

$$\tau_w = \rho_w\varkappa^2(a-m)^2\Big|\frac{\partial w}{\partial m}\Big|^2, \qquad \tau_s = f(\rho_s-\rho_w)gm\tan\varphi. \tag{3.4}$$

Here ρ is the density of the mixture of solid particles and fluid, f is the volumetric concentration of solid particles, and $p(m)$, $\tau(m)$, $\rho(m)$, and $w(m)$ are functions of the pressure, shear stress, density, and velocity of the mixture, respectively. The coordinate m directed vertically upward is reckoned from the upper boundary of the active layer: $m = 0$ on the upper boundary $z = \zeta(x,t)$, and $m = a$ on the lower boundary (see Fig. 1).

The shear stress of the mixture $\tau(m)$ is the sum of the shear stresses of the fluid $[\tau_w(m)]$ and solid $[\tau_s(m)]$ phases, which are assumed to obey the Prandtl law for turbulent flow and the Coulomb friction law, respectively.

Boundary-value conditions can be obtained from the continuity condition for the velocity and stress using the following line of reasoning [1, 2].

On the upper boundary of the layer, solid particles are in a suspended state and, hence, $\tau_s = 0$. The thickness of the layer a is small and can therefore be ignored. Hence, the quantity $\tau(0)$ is the same as that for the pure fluid at the bottom and is calculated from the solution of the hydrodynamic problem (in the absence of solid particles) $\tau^0 = \lambda\rho_w V^2$. From this for $m = 0$, we obtain the boundary conditions

$$\tau(0) = \tau^0 = \lambda\rho_w V^2;$$

$$p(0) = p^0 = \rho_w g(\eta + h - \zeta), \tag{3.5}$$

where p^0 is the pressure on the surface $z = \zeta$ which is found by solving the hydrostatic problem.

For $m \geqslant a$, the medium is at rest: $w(a) = 0$. The shear stress does not exceed the Coulomb friction: $\tau \leqslant p_s\tan\varphi$. In the moving medium ($m < a$), according to the rheological law (3.3) $\tau = p_s\tan\varphi + \tau_w$, and $\tau_w \geqslant 0$.

The continuity of τ at the point $z = a$ is possible only for $\tau_w = 0$. Thus, we obtain the condition

$$\tau(a) = \tau_s(a) = p_s \tan\varphi, \qquad p_s = (\rho_s - \rho_w)ga \tag{3.6}$$

and the attachment condition

$$w(a) = 0.$$

Condition (3.6) was considered in [16] as the criterion of the beginning of particle motion on the lower boundary of the active layer.

It should be noted that the velocity distributions in the problem considered differ significantly from that in the problem with no particles despite the identity of the turbulent friction laws. In the case of pure fluid near the wall $\tau_w \approx$ const, which implies a logarithmic velocity distribution. In the case of the mixture, τ_w increases in proportion to the distance from the wall; therefore, as $z \approx 0$, the velocity distribution is linear.

Equation (3.1) and conditions (3.5) lead to the following expressions for the pressure p in the active layer and its derivative:

$$p = \rho_w g(\eta + h - \zeta) + m\rho g, \qquad \frac{\partial p}{\partial x} = \rho_w g\Big(\frac{\partial \eta}{\partial x} - \frac{\partial \zeta}{\partial x}\Big).$$

From these expressions, we obtain

$$\frac{\partial p}{\partial x} + \rho g\,\frac{\partial \zeta}{\partial x} = \frac{\partial p}{\partial x} + (f\rho_s + \rho_w(1-f))g\,\frac{\partial \zeta}{\partial x} = \rho_w g\,\frac{\partial \eta}{\partial x} + f(\rho_s - \rho_w)g\,\frac{\partial \zeta}{\partial x}. \tag{3.7}$$

In view of (3.7), Eq. (3.2) is transformed to

$$\frac{\partial \tau}{\partial m} + \rho_w g\,\frac{\partial \eta}{\partial x} + f(\rho_s - \rho_w)g\,\frac{\partial \zeta}{\partial x} = 0. \tag{3.8}$$

Integration of Eq. (3.8) with respect to m subject to the boundary condition (3.6) yields the shear stress distribution along the thickness of the active layer

$$\tau(m) = \tau^0 - m\Big(\rho_w g\,\frac{\partial \eta}{\partial x} + f(\rho_s - \rho_w)g\,\frac{\partial \zeta}{\partial x}\Big)$$

(the shear stress on the bottom surface τ^0 should be determined from the equations of hydrodynamics). According to the adopted model (1.1), (1.2), $\tau^0 = \lambda\rho_w V^2$. Then, Eqs. (3.3) and (3.4) imply that the dependences $\tau(m)$, $\tau_s(m)$, and $\tau_w(m)$ are linear. Since $\tau_w(0) = \tau^0$ and $\tau_w(a) = 0$, it follows that

$$\tau_w(m) = \tau^0(1 - m/a). \tag{3.9}$$

Using (3.8) and boundary condition (3.6), we find the thickness of the active layer

$$a = \tau^0 \Big/ \Big[\rho_w g\,\frac{\partial \eta}{\partial x} + f(\rho_s - \rho_w)g\Big(\tan\varphi + \frac{\partial \zeta}{\partial x}\Big)\Big]. \tag{3.10}$$

Using formulas (3.4) and (3.9), we find the distribution $\partial w/\partial m$ over the thickness of the active layer

$$\frac{\partial w}{\partial m} = \frac{\sqrt{\tau_w}}{\varkappa\sqrt{\rho_w}\,(a - m)} = \sqrt{\frac{\tau^0}{\varkappa^2\rho_w a(a - m)}}. \tag{3.11}$$

It has been shown [2] that the discharge G depends weakly (and in some cases does not depend [17]) on the solid particle concentration distribution f along the depth of the active layer; therefore, the value of the parameter f can be assumed to be constant. In this case, the specific mass sediment discharge can be calculated by integration by parts, as was done in [1]:

$$G = \rho_s f \int_0^a w\,dm = -\rho_s f \int_0^a m\,\frac{\partial w}{\partial m}\,dm. \tag{3.12}$$

Substituting formula (3.11) into expression (3.12) and integrating over the thickness of the active layer, we obtain the following expression for the specific mass sediment discharge

$$G = \rho_s f a\,\frac{4}{3}\sqrt{\frac{\tau^0}{\varkappa^2\rho_w}}. \tag{3.13}$$

Finally, substitution of expression (3.10) into (3.13) yields the relation

$$G=\rho_s\,\frac{4}{3}\sqrt{\frac{\tau^0}{\varkappa^2\rho_w}}\;\tau^0 f\Big/\Big[\rho_w g\,\frac{\partial\eta}{\partial x}+f(\rho_s-\rho_w)g\tan\varphi\Big(1+\cot\varphi\,\frac{\partial\zeta}{\partial x}\Big)\Big]. \tag{3.14}$$

Assuming that the quantities $\partial\zeta/\partial x$ and $\partial\eta/\partial x$ are small, expanding Eq. (3.14) in them, and omitting terms higher than the first-order terms, we obtain

$$G=\rho_s\,\frac{4}{3}\sqrt{\frac{\tau^0}{\rho_w}}\;\frac{\tau^0}{\varkappa(\rho_s-\rho_w)g\tan\varphi}\Big(1-\cot\varphi\,\frac{\partial\zeta}{\partial x}-\frac{\rho_w\cot\varphi}{f(\rho_s-\rho_w)}\,\frac{\partial\eta}{\partial x}\Big).$$

Substituting the quadratic law $\tau^0=\rho_w\lambda V^2$ into the expression for G and linearizing it, we express the specific mass sediment discharge as

$$G=G_0U^3\Big(1+3\,\frac{\zeta-\eta}{h}-\cot\varphi\,\frac{\partial\zeta}{\partial x}-\frac{\cot\varphi}{s}\,\frac{\partial\eta}{\partial x}\Big), \tag{3.15}$$

where $s=f\gamma$ and $\gamma=(\rho_s-\rho_w)/\rho_w$.

In the active layer, the concentration of sand is $f\approx 0.1$. From (3.15) it follows that accounting for the perturbation of the free surface gives a correction with a very significant coefficient $1/s\approx 10$.

Substitution of (3.15) into (2.1) yields the linearized equation for the bed level:

$$\frac{\partial\zeta}{\partial t}+C_0\sqrt{\frac{g}{h}}\;\mathrm{Fr}^{3/2}\Big(3h\tan\varphi\Big(\frac{\partial\zeta}{\partial x}-\frac{\partial\eta}{\partial x}\Big)-h^2\,\frac{\partial^2\zeta}{\partial x^2}-\frac{h^2}{s}\,\frac{\partial^2\eta}{\partial x^2}\Big)=0. \tag{3.16}$$

Here

$$C_0=\frac{4}{3}\,\frac{\lambda^{3/2}}{\varkappa(1-\varepsilon)\gamma\tan^2\varphi}. \tag{3.17}$$

4. Linear Problem of Development of Bottom Perturbations Using the Refined Sediment Discharge Formula. Equations (1.4) and (3.16) form a closed linear problem of the development of small perturbations of the bottom and free surfaces. The perturbations of the bottom and free surfaces of the flow (ζ and η, respectively) are given by

$$\zeta=\zeta_0\exp(\sigma t+ikx),\qquad \eta=\eta_0\exp(\sigma t+ikx). \tag{4.1}$$

Substitution of (4.1) into Eqs. (1.4) and (3.16) yields

$$\begin{gathered}(ikh+3\lambda+\nu_q k^2h^2)\zeta_0+(ikh/\mathrm{Fr}-3\lambda-ikh-\nu_q k^2h^2)\eta_0=0,\\ (\sigma\sqrt{h/g}/(C_0\,\mathrm{Fr}^{3/2})+3ikh\tan\varphi+k^2h^2)\zeta_0+(k^2h^2/s-3ikh\tan\varphi)\eta_0=0.\end{gathered} \tag{4.2}$$

Setting the determinant of the homogeneous system (4.2) equal to zero, we obtain the dispersion relation

$$\frac{\sigma(k,\mathrm{Fr})}{k^2h^2C_0\,\mathrm{Fr}^{3/2}}\sqrt{\frac{h}{g}}=-1+\frac{3s\tan\varphi+(khi+3\lambda+\nu_q k^2h^2)\,\mathrm{Fr}}{s(khi(1-\mathrm{Fr})-(3\lambda+\nu_q k^2h^2)\,\mathrm{Fr})},$$

which can be represented as

$$\sigma(k,\mathrm{Fr})=\sigma_d+i\sigma_m,$$

where

$$\frac{\sigma_d}{k^2h^2C_0\,\mathrm{Fr}^{3/2}}\sqrt{\frac{h}{g}}=-\frac{1+s}{s}-\frac{s\,\mathrm{Fr}\tan\varphi\,(9\lambda+3\nu_q k^2h^2)-4k^2h^2(1-\mathrm{Fr})}{sB}; \tag{4.3}$$

$$\begin{gathered}\frac{\sigma_m}{k^3h^3C_0\,\mathrm{Fr}^{3/2}}\sqrt{\frac{h}{g}}=-\frac{(3s(1-\mathrm{Fr})\tan\varphi+\mathrm{Fr}\,(3\lambda+\nu_q k^2h^2))}{sB},\\ B=(3\lambda+\nu_q k^2h^2)^2\,\mathrm{Fr}^2+k^2h^2(1-\mathrm{Fr})^2.\end{gathered} \tag{4.4}$$

Solution (4.3), (4.4) obtained for the linearized formulation of the stability problem (4.2) allows one: to determine the regions of formation and erosion of bottom forms for various physicomechanical characteristics of bottom materials without using additional hypotheses, to determine the regions of wave distribution on the bottom surface downstream (dunes) or upstream (antidunes) versus Froude number and wavenumber k; to find the characteristic wavelengths and velocities of motion of bottom perturbations.

From (4.3) it follows that, for $\mathrm{Fr} > 1$, the minus sign at σ_m is replaced by a plus sign. In other words, the regime of dunes moving downstream is transformed to the regime of antidunes moving upstream, which is confirmed in experiments with a change in the hydrodynamic flow regime. However, at Froude numbers close, equal or greater than unity, bottom perturbations are always decaying; therefore, antidunes can exist only in a bounded time interval.

5. Analysis of the Stability Region. Bottom Perturbation Wavelengths. Valley river flows transporting sand and fine gravel, as a rule, have Froude numbers Fr = 0.01–0.10, and the flows transporting sand and fine gravel in laboratory tanks have Froude numbers Fr = 0.05–0.20 [12]. Hence, it is of interest to consider the asymptotic expansion of solution (4.3) in the small parameter Fr. To simplify the analysis of this expansion, we substitute the dimensionless wavenumber σ_d into the expression for $X = k^2h^2$. As a result, we obtain

$$\sigma_d = -C_0\,\mathrm{Fr}^{3/2}\sqrt{g/h}\,(\sigma_d^0 + \mathrm{Fr}\,\sigma_d^1 + \mathrm{Fr}^2\,\sigma_d^2 + O(\mathrm{Fr}^3)),$$

$$\sigma_d^0 = X, \qquad \sigma_d^1 = 3X\nu_q\tan\varphi + 9\lambda\tan\varphi - X/s, \tag{5.1}$$

$$\sigma_d^2 = (X\nu_q + 3\lambda)^2/s + 6\tan\varphi(3\lambda + X\nu_q) - X/s.$$

Following [12], we assume that the wavenumbers of the bottom waves are such that the initial growth rate of the amplitude has a maximum. To determine the maximum growth rate of the amplitude, we find the derivative of σ_d with respect to X and, equating it to zero,

$$\frac{\partial\sigma_d}{\partial X} = s + \mathrm{Fr}\,(3s\nu_q\tan\varphi - 1) + \mathrm{Fr}^2(2(X\nu_q^2 + 3\lambda\nu_q) + 6s\tan\varphi\nu_q - 1) = 0,$$

we determine the maximum value of the dimensionless wavenumber

$$X_{\max} = -\frac{1}{2}\,\frac{(1 + 3\nu_q\tan\varphi\,\mathrm{Fr}\,(1 + 2\,\mathrm{Fr}))s + \mathrm{Fr}(6\,\mathrm{Fr}\,\nu_q\lambda - \mathrm{Fr} - 1)}{\mathrm{Fr}^2\,\nu_q^2}. \tag{5.2}$$

Using (5.2) and substituting $X_{\max} = 4\pi^2h^2/l^2$, we obtain the asymptotic estimate of the wavelength at small Froude numbers

$$l = \frac{2\sqrt{2}\,\pi h\nu_q\,\mathrm{Fr}}{\sqrt{\mathrm{Fr}\,(1 + \mathrm{Fr}) - (1 + 3\nu_q\tan\varphi\,\mathrm{Fr}\,(1 + 2\,\mathrm{Fr}))s + 6\,\mathrm{Fr}^2\,\nu_q\lambda}}. \tag{5.3}$$

Relation (5.3) was compared with the experimental data of Hamann in the form of dependences of bottom perturbation wavelengths on the hydrodynamic flow velocity obtained in tanks (see [15]). Since Hamann did not give the value of the parameters φ, ν_t, and λ for the experiments performed, we used the values of these parameters characteristic of sandy materials: $\tan\varphi = 1/2$, $\nu_q = 0.35$, $\lambda = 0.02$, and $s = 0.1$. The theoretical and experimental data were compared for a channel of depth $h = 0.075$ m for flow velocities 0.3 m/sec $< U <$ 0.6 m/sec.

Figure 2 gives the experimental data of Hamann (points) and the calculated curve plotted using solution (5.3). Because the maximum difference between the theoretical asymptotic dependence (5.3) and the experimental data of Hamann does not exceed 6% for a systematic experimental error of 5% [15], it can be argued that the obtained asymptotic solution agrees with the experimental data not only qualitatively but also quantitatively.

It should be noted that, in the range of flow velocities 0.30 m/sec $< U <$ 0.35 m/sec, the obtained solution differ from the experimental data. This is due to the fact that the equation for the specific mass sediment discharge (3.15) and the equation of channel deformations (3.16) obtained on its basis are valid only for flow velocities far exceeding the velocity at the beginning of particle motion. Thus, at velocities close to the velocity at the beginning of particle motion, which according [15], is equal to $U = 0.28$ m/sec, calculation by formula (5.3) should give significant errors (see Fig. 2). The results show the adequacy of the mathematical model (4.2) and the correction introduced into the specific mass discharge formula (3.15).

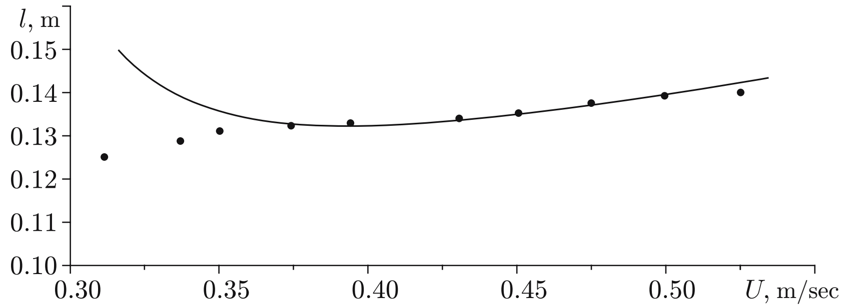

Fig. 2. Experimental (points) and calculated (curve) dependences of the critical wavelength on flow velocity ($\lambda = 0.02$ and $\nu_* = 0.35$).

We make the following remark concerning the dimensionless viscosity parameter $\nu_q = \nu_t/Q$. Using the simple algebraic model of river turbulence $\nu_t = \nu_* Q$ [14], it can be shown that the dimensionless viscosity parameter is constant

$$\nu_q = \nu_* Q/Q = \nu_*,$$

which is determined experimentally. From an analysis of function (5.1), it follows that, for Froude numbers smaller than unity, there is a spectrum of bottom wavelengths in which a region of instability exists. In the long-wave part of the spectrum, the growing perturbation wavelengths are subject to the constraint $\sigma_d < 0$, but for natural rivers and channels, there may be a more strict constraint related to the characteristic scale of river channels. The stability of short waves depends strongly on the turbulent viscosity ν_t and the hydraulic resistance coefficient λ: the lower the hydraulic resistance and flow viscosity, the shorter-wavelength perturbations of the bottom surface can arise.

Because the quantities Fr, λ, and s included in formula (5.3) are small, it can be simplified:

$$l = Kh. \tag{5.4}$$

Here

$$K = 2\sqrt{2}\,\pi\nu_q\,\mathrm{Fr}\,/\sqrt{\mathrm{Fr} + \mathrm{Fr}^2 - s(1 + 3\nu_q \tan\varphi\ \mathrm{Fr})}.$$

Formula (5.4) is similar to some well-known empirical relations [18], for example, the formulas of Snishchenko and Kopaliani

$$l = 4.2h, \tag{5.5}$$

Grishanin

$$l = (2/\lambda)^{1/3} h, \tag{5.6}$$

and Karaushev

$$l = 0.44\sqrt{1.4/\lambda + \sqrt{2/(g\lambda)}}\ h. \tag{5.7}$$

Because for the characteristic parameters s, ν_q, and φ and the Froude number $\mathrm{Fr} \simeq s(1+3\nu_q \tan\varphi\,\mathrm{Fr}) - \mathrm{Fr}^2$ specified above, the range of the coefficient K in formula (5.4) includes the entire range of the empirical multipliers at h, it can be assumed that formula (5.4) is a generalization of formulas (5.5)–(5.7).

If the condition $\mathrm{Fr} \gg s(1 + 3\nu_q \tan\varphi\,\mathrm{Fr}) - \mathrm{Fr}^2$ is satisfied, relation (5.4) can be represented as

$$l = 2\sqrt{2}\,\pi\nu_q\sqrt{\mathrm{Fr}}\,h = 2\sqrt{2}\,\pi\nu_q\sqrt{\mathrm{Fr}}\,h = 2\sqrt{2}\,\pi\nu_q U\sqrt{h/g}$$

or

$$l = K_s U\sqrt{h/g}, \tag{5.8}$$

where $K_s = 2\sqrt{2}\,\pi\nu_q$. For flow velocity far exceeding the velocity at the beginning of particle motion, formula (5.8) agrees with the formula obtained in [15] using dimension theory:

$$l = C_s(\gamma g d^3/\nu^2)^{0.1} U\sqrt{h/g}.$$

Here C_s is the experimentally determined parameter, d is the particle diameter, and ν is the fluid viscosity.

6. Determining the Propagation Velocity of Bottom Perturbations. The propagation velocity of bottom perturbations $W = -\sigma_m/k$ is determined from (4.4). Retaining only the main term of the expansion in the Froude number in (4.4), we obtain the following relation for the bottom perturbation velocity:

$$W = 3C_0 \tan\varphi \,\mathrm{Fr}^{3/2}\sqrt{gh} = 3C_0 \tan\varphi \, U \,\mathrm{Fr}\,. \tag{6.1}$$

In view of (3.17), formula (6.1) is transformed to

$$W = \frac{4\lambda^{3/2}}{\gamma\varkappa(1-\varepsilon)\tan\varphi} U \,\mathrm{Fr}\,. \tag{6.2}$$

In the case

$$\frac{4\lambda^{3/2}}{\gamma\varkappa(1-\varepsilon)\tan\varphi} = 0.0188, \tag{6.3}$$

the asymptotic formula (6.2), which is valid for flow velocity far exceeding the velocity at the beginning of particle motion, coincides with the well-known formula [19]

$$W = 0.0188 U \,\mathrm{Fr}, \tag{6.4}$$

which, in calculations, yield results in agreement with numerous experimental data [20]. We note that equality (6.3) is satisfied for standard average values of the physicomechanical parameters of the bottom material $\tan\varphi = 0.5$, $\gamma = 1$, $\varepsilon = 0.4$, $\varkappa = 0.4$, and $\lambda = 0.0068$. It is obvious, however, that for real flows, physicomechanical parameters (which are often even not detectable in experiments) can vary significantly, resulting in significant deviations from the empirical relation (6.4).

We determine the dependence of the hydraulic resistance coefficient of the flow λ on the Froude number Fr using the dependence with the exponential phenomenological parameter n given in [12, 21], taking into account only the hydraulic resistance caused by bottom waves:

$$\lambda^{3/2} = 0.0052\,\mathrm{Fr}^{3n/4}\,. \tag{6.5}$$

Then, substitution of (6.5) into (6.2) yields the dependence of the propagation velocity of bottom waves on physicomechanical parameters in general form

$$W = \frac{0.0208}{\gamma\varkappa(1-\varepsilon)\tan\varphi} U \,\mathrm{Fr}^{(3n/4)+1},$$

from which, choosing the value of the exponential coefficient n, it is possible to obtain various empirical formulas (see [12, 19–21]). Thus, the asymptotic formula (6.2), which is valid for small Froude numbers and flow velocities far exceeding the velocity at the beginning of motion of bottom particles, is a generalization of the well-known empirical formulas, which confirms the adequacy of the formulated mathematical model.

7. Modeling the Motion of a Single Perturbation at Small Froude Numbers. Using the Fourier method, we construct a solution of the problem of the development of a single bottom perturbation in time. Using formulas (1.3) and (4.3), the solutions of the problem can be represented as the Fourier integral

$$\zeta = \frac{b^2 h}{2\sqrt{\pi}} \mathrm{Re} \int\limits_{-\infty}^{\infty} \exp\Big(-\frac{b^2h^2k^2}{4}\Big) \exp\left((\sigma_d + i\sigma_m)t + ikx\right) dk. \tag{7.1}$$

At the initial time $t = 0$, the shape of the single perturbation is described by the relation

$$\zeta = \frac{b^2 h}{2\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \exp\Big(-\frac{b^2h^2k^2}{4}\Big) \cos(kx)\, dk = b\exp\Big(-\frac{x^2}{b^2h^2}\Big),$$

where b is a dimensionless coefficient that determines the amplitude of the single perturbation. Because formulas (4.3) and (4.4) defining the parameters σ_d and σ_m have a complex structure, integral (7.1) can generally be calculated only numerically. However, as noted above, valley river flows which transport sand and fine gravel, as a rule, have Froude numbers $\mathrm{Fr} = 0.01$–0.10, and the transport of sand and fine gravel in laboratory tanks requires Froude

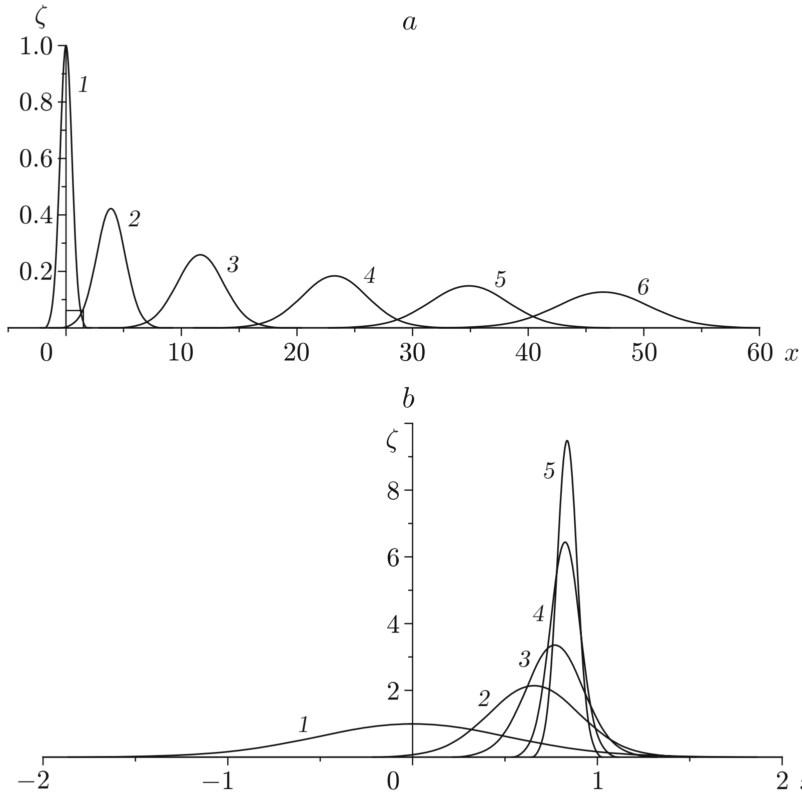

Fig. 3. Evolution of a single bottom perturbation: (a) Fr = 0.045 at $t = 0$ (1), 100 (2), 300 (3), 600 (4), 900 (5), and 1200 sec (6); (b) Fr = 0.09 at $t = 0$ (1), 6 (2), 7 (3), 7.5 (4), and 7.6 sec (5).

numbers Fr = 0.05–0.20 [12]. Hence, it is of interest to obtain an asymptotic solution of (7.1) for small Froude numbers. We expand solutions (4.3) and (4.4) in a series in small Froude numbers, retaining in them two terms of the series,

$$\sigma_d = -C_0\sqrt{g/h}\ \mathrm{Fr}^{3/2}(3s\,\mathrm{Fr}\tan\varphi\,(\nu_q k^2h^2 + 3\lambda) - k^2h^2(\mathrm{Fr} - s))/s \tag{7.2}$$

and one term of the series

$$\sigma_m = -3C_0 k\tan\varphi\ \mathrm{Fr}^{3/2}\ \sqrt{gh}. \tag{7.3}$$

Using asymptotics (7.2) and (7.3) and integrating Eq. (7.1), we obtain the solution

$$\zeta = b^2\exp\left(-9\lambda\tan\varphi\ \mathrm{Fr}\,\tau - (x/h - 3\tan\varphi\tau)^2/Z\right)/\sqrt{Z}. \tag{7.4}$$

Here $\tau = C_0\,\mathrm{Fr}^{3/2}\ \sqrt{g/h}\,t$ is dimensionless time and Z is the characteristic denominator which determines the type of evolution of a bottom wave (decay or growth):

$$Z = b^2 + 4\tau(1 + 3\nu_q\tan\varphi\ \mathrm{Fr} - \mathrm{Fr}/s). \tag{7.5}$$

Setting $3\nu_q\tan\varphi \ll 1/s$, expression (7.5) can be simplified:

$$Z = b^2 + 4\tau(1 - \mathrm{Fr}/s). \tag{7.6}$$

From (7.6), it follows that the nature of the evolution of a bottom wave is determined primarily by the mass concentration of bottom sediments s. From (7.6), it also follows that a bottom perturbation is decaying if the following condition is satisfied

$$\mathrm{Fr} < s,$$

and it is growing if

$$\mathrm{Fr} > s. \tag{7.7}$$

Of course, for $s \approx \mathrm{Fr}$, it is necessary to use the more exact stability condition

$$\mathrm{Fr} \le (1 + 3\nu_q \tan\varphi\ \mathrm{Fr})s.$$

Figure 3 shows the evolution of a single bottom perturbation for $b = 1$, $\tan\varphi = 1/2$, $s = 0.1$, $\lambda = 0.02$, $h = 1$, $C_0 = 1$, $g = 10$ m/sec^2, and various Froude numbers. It is evident that, for Fr = 0.045, the single perturbation propagating in the direction of motion of the hydrodynamic flow decays (see Fig. 3a). In this case, short-wave perturbation modes decay more rapidly than long-wave ones and the characteristic perturbation wavelength increases, resulting in an asymptotic deceleration of the erosion of the bottom perturbation. A different picture is observed if condition (7.7) is satisfied for Fr = 0.9 (see Fig. 3b). From (7.4), it follows that the perturbation growth rate is proportional to the square root of the hyperbolic function Z, i.e., it is very high. In real situations (see [10]) where the coefficient C_0 is in the range $0.0001 \le C_0 \le 0.0100$, the rate of formation of perturbations is several minutes to several hours; at the same time, the decay or establishment of bottom waves occurs much more slowly (in the range from several tens to several thousand hours).

Conclusions. The one-dimensional problem of the stability of the sandy bottom of a rectangular channel against spatially one-dimensional perturbations is formulated using the equation of the specific sediment discharge containing no empirical parameters.

The well-known sediment discharge formula is generalized by taking into account the effect of free-surface perturbations on sediment transport.

The solution of the stability problem in a linearized formulation yielded the following important results. Without using additional hypotheses and empirical parameters in the model, we obtained for the first time analytical solutions that allow one to determine the region of formation and erosion of bottom sediments for various physicomechanical characteristics of bottom materials, the region of wave propagation over the bottom surface downstream (dunes) or upstream (antidunes) versus Froude number and wavenumber.

Because of the complexity of the analytical solution obtained, it was compared with experimental data using various asymptotic approximations. A comparative analysis of the asymptotics of the obtained dependences for bottom wavelengths with the maximum initial growth rate of the amplitude showed that the obtained solution was in good agreement with experimental data.

An analysis of the analytical solution describing the propagation velocity of bottom perturbations showed that even the first term of the series in the asymptotic formula for the velocity obtained for small Froude numbers generalizes a number of well-known empirical formulas for the propagation velocity of bottom perturbations. Thus, it can be argued that the obtained asymptotic formula is supported by a great number of experimental data, which indicates the adequacy of the formulated mathematical model and the obtained analytical solution.

For small Froude numbers, an analytical solution of the problem of the evolution of a single bottom perturbation was constructed and used to derive a simple stability criterion of bottom forms.

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